Corrigé de la feuille de TD n° 4

Nombres complexes

Exercice 1

On a :

$$\mathcal{S}_1=\{z\in\mathbb{C}\mid |z|=|z-6+5i|\}=\left\{x+i\left(\frac{6}{5}x-\frac{61}{10}\right),\ x\in\mathbb{R}\right\}$$
$$\mathcal{S}_2=\{z\in\mathbb{C}\mid z(2\overline{z}+1)=1\}=\left\{-1,\frac{1}{2}\right\}$$
$$\mathcal{S}_3=\left\{z\in\mathbb{C}\mid \frac{z+4i}{5z-3}\in\mathbb{R}\right\}=\left\{x+i\left(\frac{20}{3}x-4\right),\ x\in\mathbb{R}\right\}$$
$$\mathcal{S}_4=\left\{z\in\mathbb{C}\mid \operatorname{Re}\left(\frac{z-1}{z+1}\right)=0\right\}=\{z\in\mathbb{C}\mid |z|=1\}\setminus\{-1\}$$

Exercice 2

Soit $(z,w)\in\mathbb{C}^2$. On a :

$$\begin{aligned}|z+w|^2+|z-w|^2&=(z+w)\left(\overline{z+w}\right)+(z-w)\left(\overline{z-w}\right)\\&=(z+w)\left(\overline{z}+\overline{w}\right)+(z-w)\left(\overline{z}-\overline{w}\right)\\&=z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w}+z\overline{z}-z\overline{w}-w\overline{z}+w\overline{w}\\&=2\left(|z|^2+|w|^2\right).\end{aligned}$$

Exercice 3

On a :

(E_1) $z^2+29=10z\iff z\in\{5-2i,5+2i\}$

(E_2) $z+\dfrac{1}{z}=1\iff z\in\left\{\frac{1}{2}-\frac{\sqrt{3}}{2}i,\frac{1}{2}+\frac{\sqrt{3}}{2}i\right\}$

(E_3) $z^3-3z^2+5z=3\iff z\in\left\{1,1-i\sqrt{2},1+i\sqrt{2}\right\}$

(E_4) $4z^2+12mz+36m+48=3\iff$

$$z\in\begin{cases}\left\{-\frac{3}{2}\left(m-\sqrt{(m+1)(m-5)}\right),\right.\\ \qquad\left.-\frac{3}{2}\left(m+\sqrt{(m+1)(m-5)}\right)\right\} & \text{si } m\in]-\infty,-1[\cup]5,+\infty[\\ \left\{-\frac{3}{2}m\right\} & \text{si } m=-1 \text{ ou } m=5\\ \left\{-\frac{3}{2}\left(m-i\sqrt{-(m+1)(m-5)}\right),\right.\\ \qquad\left.-\frac{3}{2}\left(m+i\sqrt{-(m+1)(m-5)}\right)\right\} & \text{si } m\in]-1,5[\end{cases}$$

(E_5) $z^7=z\iff z\in\left\{0,1,e^{i\pi/3},e^{i2\pi/3},-1,e^{i4\pi/3},e^{i5\pi/3}\right\}$

Exercice 4

Soit $(u,v)\in\mathbb{C}^2$. On a :

$$\begin{cases}u+v&=&4\\ uv&=&2(\sqrt{5}-1)\end{cases}\iff(u,v)=\left(2-\sqrt{2(3-\sqrt{5})},2+\sqrt{2(3-\sqrt{5})}\right)$$
$$\text{ou}\quad(u,v)=\left(2+\sqrt{2(3-\sqrt{5})},2-\sqrt{2(3-\sqrt{5})}\right).$$

Exercice 5

On a :

$$z_1=\left(-1+i\frac{1}{\sqrt{3}}\right)^6=\frac{64}{27}e^{i\pi}$$
$$z_2=(\cos(\theta)-i\sin(\theta))^4=e^{-4i\theta}$$
$$z_3=\frac{6-4i}{5+i}=\sqrt{2}e^{-i\pi/4}$$
$$z_4=\frac{11+3i}{1+i}=\sqrt{65}e^{-i\arccos(7/\sqrt{65})}$$

Exercice 6

Soit $(\alpha,\beta)\in\mathbb{R}^2$. On a :

$$\begin{aligned}z&=(\cos(\alpha)+\sin(\beta))+i\,(\sin(\alpha)+\cos(\beta))\\&=\left(\cos(\alpha)+\cos\left(\frac{\pi}{2}-\beta\right)\right)+i\left(\sin(\alpha)+\sin\left(\frac{\pi}{2}-\beta\right)\right)\\&=e^{i\alpha}+e^{i\left(\frac{\pi}{2}-\beta\right)}\\&=2\cos\left(\frac{\alpha-\left(\frac{\pi}{2}-\beta\right)}{2}\right)e^{i\left(\alpha+\left(\frac{\pi}{2}-\beta\right)\right)/2}\\&=2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{\pi}{4}\right)}\\&=\begin{cases}2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{\pi}{4}\right)} & \text{si } \cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)>0\\ -2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{\pi}{4}+\pi\right)} & \text{si } \cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)<0\end{cases}\\&=\begin{cases}2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{\pi}{4}\right)} & \text{si } \exists k\in\mathbb{Z},\ \frac{\alpha+\beta}{2}-\frac{\pi}{4}\in\left]-\frac{\pi}{2}+2k\pi,\frac{\pi}{2}+2k\pi\right[\\ -2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{5\pi}{4}\right)} & \text{sinon}\end{cases}\\&=\begin{cases}2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{\pi}{4}\right)} & \text{si } \exists k\in\mathbb{Z},\ \alpha+\beta\in\left]-\frac{\pi}{4}+4k\pi,\frac{3\pi}{4}+4k\pi\right[\\ -2\cos\left(\frac{\alpha+\beta}{2}-\frac{\pi}{4}\right)e^{i\left(\frac{\alpha-\beta}{2}+\frac{5\pi}{4}\right)} & \text{sinon}\end{cases}.\end{aligned}$$

Exercice 7

Linéariser les expressions suivantes pour tout $\theta \in \mathbb{R}$:

1. $\cos^2(\theta)\sin(\theta) = \frac{1}{4}(\sin(3\theta) + \sin(\theta))$
2. $\cos^2(\theta)\sin^2(\theta) = \frac{1}{8}(-\cos(4\theta) + 1)$
3. $\cos^3(\theta) = \frac{1}{4}(\cos(3\theta) + 3\cos(\theta))$
4. $\cos^2(\theta)\sin^3(\theta) = \frac{1}{16}(-\sin(5\theta) + \sin(3\theta) + 2\sin(\theta))$
5. $\cos^5(\theta) = \frac{1}{16}(\cos(5\theta) + 5\cos(3\theta) + 10\cos(\theta))$

Exercice 8

Développer les expressions suivantes pour tout $\theta \in \mathbb{R}$:

1. $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$
2. $\cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cos(4\theta) = \cos(\theta) + 2\cos(\theta)\sin(\theta) + \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$
3. $\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)$
4. $\cos(5\theta) = \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)$
5. $\sin(5\theta) = \sin^5(\theta) - 10\cos^2(\theta)\sin^3(\theta) + 5\cos^4(\theta)\sin(\theta)$